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A Counter-Example in Ring Theory and Homological Algebra*

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1. INTRODUCTION

The aim of this paper is to construct a class of principal left ideal domains (pli-domains, for short) and show that these domains provide counter-examples to some problems which were open or only partially settled.

We recall a definition. If Λ is an arbitrary ring, the transfinite powers of Λ are defined as follows: $\Lambda^1 = \Lambda$; if α is a nonlimit ordinal, say $\alpha = \beta + 1$, then $\Lambda^\alpha = \Lambda \cdot \Lambda^\beta$; if α is a limit ordinal then $\Lambda^\alpha = \bigcap_{\beta<\alpha} \Lambda^\beta$. With this definition N. Jacobson [*14*] proved the following *theorem*:
"Let R be a left Noetherian ring with Jacobson radical J. There exists an ordinal τ such that $J^\tau = (0)$". Since the ordinal τ in this theorem depends upon the ring R, there naturally arises the problem of deciding whether there exists an ordinal α which works for all left Noetherian rings. Examples of commutative local Noetherian domains show that if such an ordinal α exists, it can not be a finite ordinal. Thus the first possible candidate for such an ordinal α is ω, the first transfinite ordinal. It is well-known that $\alpha = \omega$ works for the class of commutative Noetherian rings. ([*29*]; p. 216). The following assertion is usually referred to as *Jacobson's conjecture*:

"If R is a left Noetherian ring with Jacobson radical J then $J^\omega = (0)$."

I. N. Herstein [*13*] and independently the present author [*16*] have constructed examples to show that the conjecture is not true. In [*17*], we constructed a pli-domain R in which $J^\omega \neq (0)$. However, these examples do not refute the existence of an ordinal α which would work as an index of nilpotency for the Jacobson radical of an arbitrary Noetherian ring; they only show that ω does not work. In this paper we settle the problem by showing that for every ordinal α, there exists a local pli-domain R with

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$J^{\alpha} \neq (0)$. This answers some related questions raised in ([*13*]; [*22*]; [*10*], ch. 5, p. 24).

R. E. Johnson [*19*] has considered unique factorization of elements of finite dimension in a pli-domain. He states that he does not have an example of a pli-domain containing a nonzero element of infinite dimension, although conceivably such domains exist. We construct such examples in the present paper. See also ([*8*], [*17*]). We also answer in the negative a question of P. M. Cohn [*6*] as to whether every left fir has to be a noncommutative UFD.

The problem of existence of a right primitive ring which is not left primitive was raised by N. Jacobson ([*15*], p. 4 and 255) and answered in the negative by G. M. Bergman [*2*]. We have some more examples of this kind. It may be of interest to note that our examples are pli-domains.

In [*6*], P. M. Cohn asks whether every left fir is a right fir. Subsequently, he has constructed a counter-example [*7*]. We provide some more.

H. Cartan and S. Eilenberg [*4*] had asked whether every left hereditary ring is also a right hereditary ring. I. Kaplansky [*20*] and subsequently L. W. Small ([*27*], [*28*]) and P. M. Cohn [*7*] have given examples of left hereditary rings which are not right hereditary. The greatest known difference between the left and the right global dimension of a ring appears to be 2. ([*28*], [*18*]). Now, it is well-known that if one of the left and right global dimensions is zero, then also is the other. Thus left-right asymmetry of global dimensions is possible only when both dimensions are nonzero. It is also known [*1*] that for a left Noetherian ring R,

$$\text{l. gl. dim } R \leqslant \text{r. gl. dim } R.$$

We construct examples to show the following: if $0 < m \leqslant n \leqslant \infty$, there exists a left Noetherian domain R such that $\text{l. gl. dim } R = m$ and $\text{r. gl. dim } R = n$.

In ([*12*], p. 75), I. N. Herstein states that (a statement equivalent to) the following is highly likely to be true: every non-semi-prime primary pli-ring is left and right Artinian. We have given in [*17*] a (rather uncomplicated) example to show that the conjecture is false. Taking appropriate factor rings of the local domains constructed in §4 (and denoted by P), it is easy to see that there exist non-semi-prime completely primary pli-rings containing a descending chain of two-sided ideals of an arbitrary ordinal type.

We briefly describe the layout of the paper. §3 contains the construction of required domains. In §4, we prove that the domains constructed in §3 and domains obtained from these by localization have properties required to establish our claims. §2 contains basic definitions and preliminaries needed in §3 and §4.

Our construction given in §3 is couched in the terminology of monoids. However, it is not difficult to see that we are in fact constructing certain noncommutative polynomial rings with 'binomial' relations; these relations are messy to describe even though not difficult to see through our description. The reader may find the first step given in [*17*] of some help.

As our aim in the present paper is to construct some examples, we have avoided developing results more general than immediately needed. However, it may be worthwhile to note that properties of local pli-domains P obtained in Theorems 4.6 and 4.10 hold for all local pli-domains and that the domains R of §4 have a transfinite left division algorithm. These results suggest that the domains constructed are far from unusual.

2. All rings are supposed to have unity. All subrings and ring homomorphisms are unitary. Same conventions hold for monoids (i.e., semigroups with unity.)

A domain is a ring which has no nonzero divisors. If Λ is a domain, Λ^* denotes the multiplicative monoid of nonzero elements of Λ and $U(\Lambda)$ denotes the group of units of Λ^*. Notice that in a domain Λ, $ab = 1$ implies $a, b \in U(\Lambda)$.

Suppose D is a subdomain of a domain R. Suppose x is an element of R such that every nonzero element of R can be uniquely expressed in the form

$$\sum_{i=0}^{m} d_{n_i} x^{n_i} \tag{1}$$

where m and n_i $(i = 0,..., m)$ are nonnegative integers, $0 \leqslant n_0 < \cdots < n_m$ and $d_{n_i} \in D^*$ $(i = 0,..., m)$. Further suppose, for every $d \in D$, there exists (a unique) $d' \in D$ such that

$$xd = d'x.$$

It is clear that the mapping $\rho : D \to D$ given by $\rho(d) = d'$ is a monomorphism. In the situation described above, R is called a *left twisted polynomial extension* of D, x is called *the* indeterminate in R over D and ρ is called the multiplication monomorphism determined by x. We express this in symbols as $R = D[x, \rho]$. It is clear that the choice of x need not be unique. However, we shall have no occasion to change indeterminates and shall treat x and ρ as if they are uniquely determined by R and D.

The unique expression of the form (1) for a nonzero element of R is called its *standard form*. The standard form of 0 is assumed to be 0. The *degree* of an element of R can be defined in an obvious manner using (1). It is clear the

$$\deg fg = \deg f + \deg g \tag{2}$$

holds for every $f, g \in R$.

Domains of the type $D[x, \rho]$ were considered by Ore [*23*].

Let D be a domain and M be a submonoid of D^*. M is said to be a *left localizable monoid in* D if for every $(a, d) \in M \times D^*$ there exists $(a_1, d_1) \in M \times D^*$ such that

$$a_1 d = d_1 a. \tag{3}$$

A domain D is called a left Ore domain if D^* is a left localizable monoid in D.

THEOREM 2.1. *If M is a left localizable monoid in a domain D then there exists a domain $\tilde{D}$ containing D as a subdomain such that $M \subseteq U(\tilde{D})$ and every element of $\tilde{D}$ can be expressed (in at least one way) as $a^{-1}d$ with $a \in M$, $d \in D$. Further, $\tilde{D}$ is uniquely determined by D and M upto an isomorphism which is trivial on D.*

If $\sigma : D \to D$ is a monomorphism with $\sigma(M) \subseteq M$ then σ can be uniquely extended to a monomorphism $\tilde{\sigma} : \tilde{D} \to \tilde{D}$.

Proof. The construction of $\tilde{D}$ is a modification of the corresponding commutative situation cf. ([*29*], p. 47). (3) is used for getting 'common denominators'. For details, see ([*3*], p. 162). The other assertion is trivial.

We shall denote $\tilde{D}$ by D_M. If D is a left Ore domain, D_{D^*} is clearly a skew field called the left quotient skew field of D.

PROPOSITION 2.2. *Let M be a left localizable monoid in a domain D. For arbitrary $a_1, \ldots, a_n \in M$, there exist $d_1, \ldots, d_n \in D^*$ and $a \in M$ such that*

$$a = d_i a_i \qquad (i = 1, \ldots, n).$$

Proof. For $n = 1$, the proposition is clear. Assume that $\bar{d}_1, \ldots, \bar{d}_{n-1} \in D^*$ and $\bar{a} \in M$ exist such that $\bar{a} = \bar{d}_i a_i$, $i = 1, \ldots, n-1$. There exist $b \in M$, $d_n \in D^*$ such that $b\bar{a} = d_n a_n$; put $a = b\bar{a}$ and observe that $a \in M$. Set $d_i = b\bar{d}_i$ for $i = 1, \ldots, n-1$. This completes the induction and concludes the proof.

PROPOSITION 2.3. *Let $R = D[x, \rho]$ be a domain. Let M be a left localizable monoid in D such that $\rho(M) \subseteq M$. Then M is a left localizable monoid in R.*

Proof. We have to show that for every $a \in M$ and $g = \Sigma d_i x^i \in R^*$ $(d_i \in D)$, there exist $a' \in M$, $h = \Sigma d_i' x^i \in R^*$ $(d_i' \in D)$ such that $a'g = ha$. Degree formula (2) shows that $\deg g = \deg h = n$ say. A simplification using $xd = \rho(d)x$ shows that

$$a'd_i = d_i' \rho^i(a) \tag{4}$$

must hold for $i = 0, \ldots, n$. It thus suffices to prove existence of $a' \in M$, $d_i' \in D$ $(i = 0, \ldots, n)$ satisfying (4).

Since $\rho(M) \subseteq M$ and M is left localizable in D, there exist $a_0', ..., a_n' \in M$ and $c_0, ..., c_n \in D$ such that

$$a_i'd_i = c_i\rho^i(a), \qquad i = 0, ..., n.$$

By Proposition 2.2, there exist $b_0, ..., b_n \in D$ and $a' \in M$ such that $a' = b_i a_i'$, $i = 0, ..., n$. Thus equations (4) hold with $d_i = b_i c_i$, $i = 0, ..., n$. This proves the proposition.

PROPOSITION 2.4. *$R = D[x, \rho]$ is a left Ore domain if D is a left Ore domain.*

Proof. Since D^* is left localizable in D and $\rho(D^*) \subseteq D^*$, Proposition 2.3 shows that $M = D^*$ is left localizable in R. Let L be the left quotient skew field of D and $\tilde{\rho} : L \to L$ be the unique extension of ρ to L. It is then easy to see that $R_M \simeq L[x, \tilde{\rho}]$. It is well-known (and easy to see) that $L[x, \tilde{\rho}]$ is a left Ore domain. Let $r_1, r_2 \in R^*$. As R_M is left Ore, there exist $\alpha, \beta \in R_M^*$ such that $\alpha r_1 = \beta r_2$. However, we may express α, β as $a^{-1}f$, $a^{-1}g$ respectively where $a \in D^*$ and $f, g, \in R^*$. Thus $fr_1 = gr_2$ and R is left Ore domain.

For an alternative proof, see [9].

A domain D is a principal left ideal domain, *pli-domain* for short, if every left ideal is of the form Dd, $d \in D$.

THEOREM 2.5. *$R = D[x, \rho]$ is a pli-domain if and only if D is a pli-domain and $\rho(D^*) \subseteq U(D)$.*

Proof. Suppose R is a pli-domain. Clearly Rx is an ideal of R and $R/Rx \simeq D$ so that D is a pli-domain.

Let $d \in D^*$ be an arbitrary element. Then

$$Rd + Rx = Rf$$

holds for some $f \in R^*$. The degree rule (2) shows that $f \in D^*$ since $d \in Rf \cap D^*$. Also, $x = gf$ for some $g \in R$ with $\deg g = 1$. Let $g = g_1x + g_0$, $g_1, g_0 \in D$, $g_1 \neq 0$. Then

$$x = gf = g_1xf + g_0f = g_1\rho(f)x + g_0f.$$

The assumed uniqueness of form (1) in R shows that

$$1 = g_1\rho(f).$$

Since $f \in Rd + Rx$, $f = Fd + Gx$ for some $F, G \in R$. Comparing constant terms in the standard form of both sides, we get

$$f = a'd, \qquad a' \in D^*.$$

Consequently,

$$1 = g_1\rho(a')\,\rho(d)$$

and $\rho(d) \in U(D)$. Thus $\rho(D^*) \subseteq U(D)$.

Conversely suppose D is a pli-domain and $\rho(D^*) \subseteq U(D)$. Let l be a non-zero left ideal of R, n the least degree of nonzero elements in l, L the subset of D consisting of zero together with the leading coefficients of elements of degree n in l when written in the standard form. Clearly, L is a non-zero left ideal of D, say $D\bar{d} = L$. Let f be an element of degree n in l with leading coefficient $\bar{d}$ in the standard form. Since $[\rho(\bar{d})]^{-1}\,xf$ is a monic in its standard form, and has degree $n + 1$, it follows that for every $g \in l$, there exists $h \in R$ such that $\deg(g - hf) \leqslant n$. Since every polynomial of degree n in l is easily seen to be in Rf, we have $l = Rf$. This completes the proof.

We shall require the following generalization of left twisted polynomial extensions.

Suppose D is a subdomain of a domain R. For some ordinal $\alpha > 1$, suppose $\{R_\beta : \beta < \alpha\}$ is a set of subdomains of R satisfying the following conditions:

(i) $D = R_0$; $R = \bigcup_{\beta<\alpha} R_\beta$.

(ii) for $0 < \beta < \alpha$, $R_\beta = (\bigcup_{\gamma<\beta} R_\gamma)[x_\beta\,, \rho_\beta]$.

Then R is called a *generalized left twisted polynomial extension* of D; $\{R_\beta : \beta < \alpha\}$ is called *the* chain of twisted subdomains from D to R and $X = \{x_\beta : 0 < \beta < \alpha\}$ is called *the* set of indeterminates in R over D. Again, we shall ignore the possibility of changing indeterminates and treat them as if they are uniquely determined by R and D. The situation shall be expressed symbolically as $R = D[x_\beta\,, \rho_\beta : 0 < \beta < \alpha]$. For convenience, we put $\bar{R}_\beta = \bigcup_{\gamma<\beta} R_\gamma = D[x_\gamma\,, \rho_\gamma : 0 < \gamma < \beta]$.

We can introduce a standard form for elements of

$$R = D[x_\beta\,, \rho_\beta : 0 < \beta < \alpha].$$

As usual, a product over an empty indexing set shall stand for 1. A nonzero element of R is in the standard form when it is expressed as

$$\sum_{i=0}^{m} d_i x_{\alpha_{i1}} \cdots x_{\alpha_{in_i}}$$

where m and n_i $(i = 0,\ldots, m)$ are nonnegative integers, $d_i \in D^*$ $(i = 0, \ldots, m)$, α_{ij} $(j = 1,\ldots, n_i)$ are ordinals such that $0 < \alpha_{i1} \leqslant \cdots \leqslant \alpha_{in_i}$ for every i and the sequences $(\alpha_{i1}\,,\ldots, \alpha_{in_i})$ are not identical for distinct values of i. The standard form of 0 is assumed to be 0.

Extensions of the above type have been considered by C. W. Curtis [9].

THEOREM 2.6. *Every element of* $D[x_\beta, \rho_\beta : 0 < \beta < \alpha]$ *has a unique standard form.*

Proof. The assertion is clear for a left twisted polynomial extension of a domain. An easy transfinite induction suffices to conclude the proof.

THEOREM 2.7. *Let* $R = D[x_\beta, \rho_\beta : 0 < \beta < \alpha]$ *be a domain. If* M *is a left localizable monoid in* D *and if* $\rho_\beta(M) \subseteq M$ *for* $0 < \beta < \alpha$ *then* M *is a left localizable monoid in* R.

Proof. An easy transfinite induction using Proposition 2.3 suffices.

THEOREM 2.8. $R = D[x_\beta, \rho_\beta : 0 < \beta < \alpha]$ *is a left Ore domain if* D *is a left Ore domain.*

Proof. An easy transfinite induction using Proposition 2.4 suffices.
See [9] for an alternative proof.

Recall that $\bar{R}_\beta$ stands for $\bigcup_{\gamma<\beta} R_\gamma$.

LEMMA 2.9. $R = D[x_\beta, \rho_\beta : 0 < \beta < \alpha]$ *is a pli-domain if each* $R_\beta = D[x_\gamma, \rho_\gamma : 0 < \gamma \leq \beta]$ *is a pli-domain for* $\beta < \alpha$.

Proof. The lemma is trivial if α is a nonlimit ordinal. Let I be a nonzero left ideal of R. Let $\lambda < \alpha$ be an ordinal such that $I \cap R_\lambda \neq (0)$, say $I \cap R_\lambda = R_\lambda f, f \in R_\lambda^*$ (as R_λ is a pli-domain). We shall show that $I = Rf$. Clearly $I \supseteq Rf$. To prove the other inclusion, it suffices to show that $x_\mu \in Rf$ for $\lambda < \mu < \alpha$; for then using the standard form of elements in R, it can be readily seen that $I \subseteq Rf$. Let $\lambda < \mu < \alpha$. Then $R_\mu = \bar{R}_\mu[x_\mu, \rho_\mu]$ and $f \in \bar{R}_\mu^*$. By Theorem 2.5, $\rho_\mu(f) \in U(\bar{R}_\mu)$, and

$$x_\mu = [\rho_\mu(f)]^{-1} x_\mu f \in Rf.$$

The lemma is thus proved.

THEOREM 2.10. $R = D[x_\beta, \rho_\beta : 0 < \beta < \alpha]$ *is a pli-domain if and only if* D *is a pli-domain and* $\rho_\beta(\bar{R}_\beta^*) \subseteq U(D)$ *for* $0 < \beta < \alpha$.

Proof. If D is a pli-domain and $\rho_\beta(\bar{R}_\beta^*) \subseteq U(D)$ for $0 < \beta < \alpha$ then a transfinite induction on β using Theorem 2.5 and Lemma 2.9 shows that R is a pli-domain.

Suppose R is a pli-domain. Clearly, R is also a generalized left twisted polynomial extension of R_β; in fact,

$$R = R_\beta[x_\lambda, \rho_\lambda : \beta < \lambda < \alpha].$$

It is easy to see that the left ideal I of $R_\beta[x_\lambda, \rho_\lambda : \beta < \lambda < \alpha]$ generated by the set $\{x_\lambda : \beta < \lambda < \alpha\}$ is an ideal of R and $R/I \simeq R_\beta$. Thus every R_β, $(\beta < \alpha)$, is a pli-domain. Since $R_\beta = \bar{R}_\beta[x_\beta, \rho_\beta]$ for $0 < \beta < \alpha$, it follows from Theorem 2.5 that $\rho_\beta(R_\beta^*) \subseteq U(\bar{R}_\beta)$. However it is easy to see that $U(\bar{R}_\beta) = U(D)$. This completes the proof.

3. In this section, we give a construction to prove the following theorem.

THEOREM 3.1. *Let k be an arbitrary skew field and let $\alpha > 1$ be an arbitrary ordinal. There exists a skew field K containing k as a subskew field and a domain $R = K[x_\beta, \rho_\beta : 0 < \beta < \alpha]$ such that for $0 < \beta < \alpha$,*

$$\rho_\beta(\bar{R}_\beta) \subseteq K.$$

Further, if $|k| \leqslant \aleph_0$ then R may be so chosen that $|R| = \aleph_0 |\alpha|$.

The proof is carried out in two steps. Firstly we construct a special monoid M_α for each ordinal α. We then take the monoid-ring $k(M_\alpha)$ of M_α over k and show that a certain submonoid N of $k(M_\alpha)^*$ is left localizable in $k(M_\alpha)$. The required ring R turns out to be $k(M_\alpha)_N$. The cardinality restriction is easy to obtain.

We recall a few things about monoids. A monoid M is cancellative if $ab = ac$ or $ba = ca$ implies $b = c$. If $\{M_i : i \in I\}$ is a non-empty family of monoids then $\prod_{i\in I}^{w} M_i$ denotes their weak direct product (i.e., for every element, all but a finite number of coordinates are unity and multiplication is coordinatewise.) $\psi_i : M_i \to \prod_{i\in I}^{w} M_i$ denotes the ith canonical injection. The set of endomorphisms (resp. mono-endomorphisms) of a monoid M is itself a monoid under composition as the multiplication; this monoid is denoted by $\mathscr{E}(M)$ (resp. $\mathscr{M}\mathscr{E}(M)$).

We shall require an analogue of the construction of a semi-direct product of groups. Cf. ([11]; p. 88). Let M and M' be monoids and let $\varphi : M' \to \mathscr{E}(M)$ be a homomorphism. The semi-direct product of M with M' by φ, denoted by $M \times_\varphi M'$, is the monoid on $M \times M'$ with the following rule of multiplication:

$$(a, b)(a_1, b_1) = (a\varphi_b(a_1), bb_1)$$

where $a, a_1 \in M$; $b, b_1 \in M'$ and $\varphi_b = \varphi(b)$. It is easy to see that $M \times_\varphi M'$ is cancellative if and only if M and M' are cancellative and $\varphi(M') \subseteq \mathscr{M}\mathscr{E}(M)$.

The following definition formulates the special kind of monoids we need.

DEFINITION Let M be a concellative monoid and $B = \{b_\lambda : 0 < \lambda < \tau\}$ be a set of generators of M indexed by the set of all nonzero ordinals less than a certain ordinal τ. Let $<$ be the well-order on B induced by the well-order on the set $\{\lambda : 0 < \lambda < \tau\}$. $(B, <)$ is called a *basis* of M if the following conditions are satisfied:

(B1) Every element of M can be uniquely expressed in the form

$$b_{\alpha_1} \cdots b_{\alpha_n}$$

where $0 < \alpha_1 \leqslant \cdots \leqslant \alpha_n < \tau$.

(B2) For $0 < \beta < \lambda < \tau$, there exists $0 < \gamma < \lambda$ such that

$$b_\lambda b_\beta = b_\gamma b_\lambda .$$

The following lemma is clear.

LEMMA 3.2. (a) *Let I be a non-empty well-ordered set. Let M be a monoid and $M = \bigcup_{i \in I} M_i$ where each M_i is a submonoid of M having a basis $(B_i, <_i)$. Let $(B_i, <_i)$ be an initial segment of $(B_j, <_j)$ whenever $i, j \in I$, $i < j$. Let $B = \bigcup_{i \in I} B_i$ be given the unique well-order $<$ which is compatible with $<_i$ on B_i for each $i \in I$. Then $(B, <)$ is a basis of M.*

(b) *Let $\{M_n : n \in Z^+\}$ be a family of monoids. Let $(B_n, <_n)$ be a basis of M_n for each $n \in Z^+$. Then $\prod^w_{n \in Z^+} M_n$ has $B = \bigcup_{n \in Z^+} \psi_n(B_n)$ as a basis where B is given the well-order of a well-ordered sum of well-ordered sets.*

(c) *Let $(B, <)$ and $(B', <')$ be bases of M and M' respectively. Let $\varphi : M' \to \mathcal{M}\mathcal{E}(M)$ be a homomorphism such that $\varphi_x(B) \subseteq B$ for every $x \in M'$. Let $\bar{B} = \{(B, e')\} \cup \{(e, B')\}$ be given the well-order of a sum. Then $\bar{B}$ is a basis of $M \times_\varphi M'$.*

We now begin the construction needed to prove theorem 3.1. The following lemma forms the first step in the construction.

LEMMA 3.3. *Let $\alpha > 1$ be an arbitrary ordinal. There exist ordinals η and θ, a monoid M with $| M | = \aleph_0 | \alpha |$ and a basis $B = \{b_\beta : 0 < \beta < \theta\}$ of M such that the set $\{\xi : \eta \leqslant \xi < \theta\}$ is order-isomorphic to $\{\xi : 0 < \xi < \alpha\}$ and the following condition holds:*

(B3) *If $0 < \beta < \lambda < \theta$ and $\eta \leqslant \lambda$ then there exist $0 < \gamma < \eta$ such that*

$$b_\lambda b_\beta = b_\gamma b_\lambda .$$

It may be worthwhile to note the difference between (B2) and (B3). As our monoid has a basis, (B1) and (B2) assert that given β, λ with $\beta < \lambda$, there exists a *unique* γ with $\gamma < \lambda$ such that $x_\lambda x_\beta = x_\gamma x_\lambda$. (B3) asserts more than the inequality $\gamma < \lambda$ if β happens to belong to the tail $\{\xi : \eta \leqslant \xi < \theta\}$. We shall see that this tail plays a crucial role in the second stage of our construction.

Proof. Let L_1 be the free monoid on a singleton set $\{b_1\} = B_1$. Clearly B_1 with the trivial well-order is a basis of L_1.

We now describe an iterative procedure. Suppose, for some ordinal $\lambda > 1$, the set $\{L_{\lambda'} : 0 < \lambda' < \lambda\}$ of monoids is obtained by iteratively applying the procedure described below to L_1. Then it happens that

(T1) each $L_{\lambda'}$ has a basis $(B_{\lambda'}, <_{\lambda'})$.

(T2) if $0 < \lambda_1 < \lambda_2 < \lambda$ then L_{λ_1} is a submonoid of L_{λ_2} and $(B_{\lambda_1}, <_{\lambda_1})$ is an initial segment of $(B_{\lambda_2}, <_{\lambda_2})$.

Put

$$\bar{L}_\lambda = \bigcup_{0<\lambda'<\lambda} L_{\lambda'}$$

and

$$\bar{B}_\lambda = \bigcup_{0<\lambda'<\lambda} B_{\lambda'} .$$

Then there is a unique monoid structure on $\bar{L}_\lambda$ which is compatible with the monoid structure on $L_{\lambda'}$ for each λ', $0 < \lambda' < \lambda$. Also, there is a unique well order, say $\bar{<}_\lambda$, on $\bar{B}_\lambda$ which is compatible with $<_{\lambda'}$ on $B_{\lambda'}$ for each λ', $0 < \lambda' < \lambda$. By Lemma 3.2 (a), $(\bar{B}_\lambda, \bar{<}_\lambda)$ is a basis of $\bar{L}_\lambda$. Let

$$L_\lambda^* = \prod_{n\in Z^+}^{w} \bar{L}_{\lambda n}$$

where each $\bar{L}_{\lambda n}$ is the monoid $\bar{L}_\lambda$. Identify $\bar{L}_\lambda$ with $\psi_1(\bar{L}_{\lambda 1})$. Let

$$B_\lambda^* = \bigcup_{n\in Z^+} \psi_n(\bar{B}_\lambda).$$

Because of our identification, $\psi_1(\bar{B}_\lambda) = \bar{B}_\lambda$. Let $<_\lambda^*$ be the unique well-order on B_λ^* defined as a well-ordered sum of well-ordered sets. By Lemma 3.2 (b), $(B_\lambda^*, <_\lambda^*)$ is a basis of L_λ^*.

Define a mapping

$$g_\lambda : L_\lambda^* \to L_\lambda^*$$

as follows: every element of L_λ^* is a sequence $(a_n)_{n\in Z^+}$ and $a_n = 1$ in $\bar{L}_\lambda$ for all but finitely many n. Put

$$g_\lambda((a_n)) = (a_n')$$

where $a_1' = 1$ and $a_{n+1}' = a_n$ for every $n \in Z^+$. It is clear that g_λ is a mono-endomorphism of L_λ^* and $g_\lambda(B_\lambda^*) \subseteq B_\lambda^*$. Let b_λ be an element (of a larger set) which is not in L_λ^*. Let F_λ be the free monoid on the singleton set $\{b_\lambda\}$. Let

$$\varphi_\lambda : F_\lambda \to \mathcal{ME}(L_\lambda^*)$$

be the homomorphism defined by $b_\lambda{}^n \to g_\lambda{}^n$ for $n = 0, 1, \ldots$ Define

$$L_\lambda = L_\lambda^* \times {}_{\varphi\lambda}F_\lambda .$$

Identify the first coordinate monoid of L_λ with L_λ^* . Define

$$B_\lambda = B_\lambda^* \cup \{b_\lambda\}$$

and define a well-order $<_\lambda$ on B_λ as follows: on B_λ^* , $<_\lambda^*$ and $<_\lambda$ agree; b_λ is the least element of B_λ . Lemma 3.2 (c) shows that $(B_\lambda , <_\lambda)$ is a basis of L_λ . The various identifications made in the above procedure show that (T1) and (T2) hold for all λ', $0 < \lambda' \leq \lambda$. This shows that our procedure can be iterated at any nonlimit ordinal stage.

Suppose λ_0 is a limit ordinal and suppose we have obtained sets $\{L_{\lambda'} : 0 < \lambda' < \lambda\}$ of monoids satisfying (T1) and (T2) for every $\lambda < \lambda_0$. Then it is clear that the set $\{L_{\lambda'} : 0 < \lambda' < \lambda_0\}$ of monoids also satisfies (T1) and (T2). Thus our procedure can be iterated at any limit ordinal stage also.

Since we have explicitly given L_1 with a basis B_1 and since we have shown that our procedure enables us to construct a specific monoid L_λ with a basis $(B_\lambda , <_\lambda)$ in terms of $\{L_{\lambda'} : 0 < \lambda' < \lambda\}$ with bases $(B_{\lambda'} , <_{\lambda'})$, we have obtained a monoid L_λ with a basis $(B_\lambda , <_\lambda)$ for every ordinal $\lambda > 0$.

We emphasize that L_λ is not just a monoid with basis obtained from $\{L_{\lambda'} : 0 < \lambda' < \lambda\}$ by a specific procedure but each $L_{\lambda'}$, $\lambda' > 1$ is itself obtained from earlier ones by the same procedure. As the structure of L_1 is clear and as our procedure is iterative and not hard to see through, we know a set of defining relations for L_λ for the generating set B_λ .

Now, let $\alpha > 1$ be the ordinal stated in the lemma. If α is a nonlimit ordinal, say $\alpha = \lambda + 1$, then let

$$M = L_\lambda \qquad \text{and} \qquad (B, <) = (B_\lambda , <_\lambda).$$

If α is a limit ordinal, let

$$M = \bar{L}_\alpha \qquad \text{and} \qquad (B, <) = (\bar{B}_\alpha , \bar{<}_\alpha).$$

As $\aleph \cdot \aleph = \aleph \cdot \aleph_0 = \aleph$ holds cf. ([*26*], p. 392), we get $| M | = \aleph_0 \cdot | \alpha |$.

Observe that for each λ, $(0 < \lambda < \alpha)$, $(B_\lambda , <_\lambda)$ is an initial segment of $(B, <)$ and $(B_\lambda , <_\lambda)$ has b_λ as the last element. Let $I = \{b_\lambda : 0 < \lambda < \alpha\}$ i.e., I is the subset of $(B, <)$ consisting of the last element of $(B_\lambda , <_\lambda)$ for every $0 < \lambda < \alpha$. We need the following

SUBLEMMA: *If $b_\lambda \in I$, $b \in B - I$ and $b_\lambda < b$ as elements of $(B, <)$ then*

$$bb_\lambda = b_\lambda b. \tag{5}$$

Proof of the Sublemma. Let $\xi = \min\{\zeta : b \in B_\zeta\}$. Clearly $\lambda < \xi$ since b_λ is the last element of $(B_\lambda, <_\lambda)$ and $b \notin I$. As $b \in B_\xi = B_\xi^* \cup \{b_\xi\}$ and $b \notin I$, we have $b \in B_\xi^*$. Now $B_\xi^* = \bigcup_{n \in Z^+} \psi_n(\bar{B}_\xi)$ and $\psi_n(\bar{B}_\xi)$ are mutually disjoint. Thus $b \in \psi_m(\bar{B}_\xi)$ for precisely one positive integer m. If $m = 1$ then due to our identifications, $b \in \psi_1(\bar{B}_\xi) = \bar{B}_\xi = \bigcup_{\mu < \xi} B_\mu$ so that $b \in B_\mu$ for some $\mu < \xi$. However, this contradicts our choice of ξ. Thus $b \in \psi_m(\bar{B}_\xi)$ for precisely one integer $m > 1$. As $\lambda < \xi$, $b_\lambda \in \bar{B}_\xi = \psi_1(\bar{B}_\xi)$. Clearly, b and b_λ commute as elements of L_ξ^*. As L_ξ^* is a submonoid of M, the sublemma is proved.

We are now in a position to finish the proof of Lemma 3.3. We define a new well-order say $\dashv$ on B as follows: On I and $B - I$, take $\dashv$ to be the restriction of $<$ but make every element of I follow every element of $B - I$ in the order $\dashv$. As $\dashv$ is defined as a well-ordered sum of two well-ordered sets, $\dashv$ is a well-order on B. Using (5), it is easy to see that $(B, \dashv)$ is a basis of M.

Let $(B, \dashv)$ be indexed by the set of all nonzero ordinals less than a certain ordinal θ. Let η be the subscript of b_1 in this new indexing where b_1 is the only element in the basis B_1 of L_1. The lemma clearly holds for these values of θ and η. This completes the proof of lemma 3.3.

The next lemma relates monoids with bases to generalized left twisted polynomial extensions. The following notation will be convenient. Suppose M as an arbitrary monoid with a basis $B = \{b_\beta : 0 < \beta < \theta\}$. Then M^γ denotes the submonoid of M generated by $\{b_\beta : 0 < \beta \leqslant \gamma\}$ and $\bar{M}^\gamma = \bigcup_{\mu < \gamma} M^\mu$. Also $M^0 = \{1\}$. It is clear that M^γ and $\bar{M}^\gamma$ have $\{b_\beta : 0 < \beta \leqslant \gamma\}$ and $\{b_\beta : 0 < \beta < \gamma\}$ as bases respectively.

If Λ is a ring and M is an arbitrary monoid, the monoid ring of M over Λ is denoted by $\Lambda(M)$. Recall that the additive group of $\Lambda(M)$ is the free left Λ-module of formal Λ-linear combinations of elements of M. The multiplication in $\Lambda(M)$ is defined by identifying the unity of Λ with the unity of M, assuming the distributive laws and the rule

$$(\lambda_1 a_1)(\lambda_2 a_2) = \lambda_1\lambda_2\, a_1 a_2$$

for $\lambda_i \in \Lambda$, $a_i \in M$, $i = 1, 2$.

LEMMA 3.4. *Let Λ be a domain and M be a monoid with a basis $B = \{b_\beta : 0 < \beta < \theta\}$. Then $\Lambda(M)$ is a generalized left twisted polynomial extension of Λ with a chain $\{\Lambda(M^\beta) : \beta < \theta\}$ of twisted subdomains from Λ to $\Lambda(M)$. B is a set of indeterminates in $\Lambda(M)$ over Λ.*

Proof. $\Lambda(M^1)$ is clearly a polynomial ring in one commuting indeterminate b_1 over Λ. Thus $\Lambda(M^1)$ is a generalized left twisted polynomial extension of Λ with $\{\Lambda(M^0), \Lambda(M^1)\}$ as a chain of twisted subdomains from Λ to $\Lambda(M^1)$.

Further, $\{b_1\}$ is a basis of M^1 and also a set of indeterminates in $\Lambda(M^1)$ over Λ.

We now proceed by a transfinite induction. For each γ, $\gamma < \beta$, suppose $\Lambda(M^\gamma)$ is a generalized left twisted polynomial extension of Λ with $\{\Lambda(M^\tau) : \tau \leqslant \gamma\}$ as a chain of twisted subdomains from Λ to $\Lambda(M^\gamma)$. Further, suppose $\{b_\tau : 0 < \tau \leqslant \gamma\}$ is a set of indeterminates in $\Lambda(M^\gamma)$ over Λ. Then $\bigcup_{\gamma<\beta} \Lambda(M^\gamma) = \Lambda(\bar{M}_\beta)$ is clearly a generalized left twisted polynomial extension of Λ with a chain $\{\Lambda(M^\gamma) : \gamma < \beta\}$ of twisted subdomains from Λ to $\Lambda(\bar{M}^\beta)$. Also, $\{b_\gamma : 0 < \gamma < \beta\}$ is a set of indeterminates in $\Lambda(\bar{M}^\beta)$ over Λ.

Let $a \in \bar{M}^\beta$ be an arbitrary element. Since $\{b_\lambda : 0 < \lambda < \beta\}$ is a generating set (in fact, a basis) of $\bar{M}^\beta$, we can repeatedly use (*B2*) and get an element $a' \in \bar{M}^\beta$ such that

$$b_\beta a = a' b_\beta .$$

Since M is cancellative, a' is uniquely determined by a. It is clear that the mapping $a \mapsto a'$ is a mono-endomorphism of $\bar{M}^\beta$; it can thus be uniquely extended to a ring monomorphism say $\rho_\beta : \Lambda(\bar{M}^\beta) \to \Lambda(\bar{M}^\beta)$ which is trivial on Λ. In $\Lambda(M^\beta)$, we now have

$$b_\beta c = \rho_\beta(c)\, b_\beta$$

for every $c \in \Lambda(\bar{M}^\beta)$. It is now clear that

$$\Lambda(M^\beta) = \Lambda(\bar{M}^\beta)[b_\beta , \rho_\beta].$$

This completes the transfinite induction. It is now easy to complete the proof.

The following lemma is the second stage of our construction.

LEMMA 3.5. *Let k be an arbitrary skew field. Let M be a monoid with a basis $\{b_\beta : 0 < \beta < \theta\}$ and let η be an ordinal such that* (B3) *is satisfied in M. Let $\tilde{M} = \bar{M}^\eta$ and $N = k(\tilde{M})^*$. Then N is a left localizable monoid in $k(M)$. Let $R = k(M)_N$. Then R contains a left quotient skew field K of $k(\tilde{M})$ and R has the form*

$$R = K[b_\beta , \rho_\beta : \eta \leqslant \beta < \theta]$$

where

$$\rho_\beta(K[b_\gamma , \rho_\gamma : \eta \leqslant \gamma < \beta]) \subseteq K$$

for every $\beta, \eta \leqslant \beta < \theta$.

Proof. As we have already remarked, $\tilde{M}$ has $\{b_\beta : 0 < \beta < \eta\}$ as a basis. Theorem 2.8 and Lemma 3.4 show that $k(\tilde{M})$ and $k(M)$ are left Ore domains. It is easy to see that

$$k(M) = k(\tilde{M})[b_\beta , \rho_\beta : \eta \leqslant \beta < \theta].$$

Condition (B3) shows that for every β with $\eta \leqslant \beta < \theta$, we have $\rho_\beta(N) \subseteq N$. Since $k(\tilde{M})$ is a left Ore domain, N is a left localizable monoid in $k(\tilde{M})$. Theorem 2.7 now shows that N is a left localizable monoid in $k(M)$. We can thus form the domain $R = k(M)_N$. Clearly,

$$K = \{a^{-1}b \mid a, b \in k(\tilde{M});\ a \neq 0\}$$

is a subskew field of R which is also a left quotient skew field of $k(\tilde{M})$. It is evident that $b_\beta \notin K$ for $\eta \leqslant \beta < \theta$.

For $\eta \leqslant \beta < \theta$, let

$$D_\beta = k(\tilde{M})[b_\gamma, \rho_\gamma : \eta \leqslant \gamma \leqslant \beta]$$

and

$$\bar{D}_\beta = k(\tilde{M})[b_\gamma, \rho_\gamma : \eta \leqslant \gamma < \beta].$$

Since N is left localizable in $k(\tilde{M})$ and $\rho_\beta(N) \subseteq N$ for $\eta \leqslant \beta < \theta$, it follows from Theorem 2.7 that N is a left localizable monoid in each D_β and $\bar{D}_\beta$, $\eta \leqslant \beta < \theta$. Also, by Theorem 2.1, ρ_β can be uniquely extended to a mono-endomorphism of $(\bar{D}_\beta)_N$; We shall denote the extended map also as ρ_β. Then $\rho_\beta((\bar{D}_\beta)_N) \subseteq K$ as is seen by using (B3). Now $\{K\} \cup \{(D_\beta)_N : \eta \leqslant \beta < \theta\}$ is a chain of twisted subdomains from K to R, $(D_\beta)_N = (\bar{D}_\beta)_N[b_\beta, \rho_\beta]$ and $R = K[b_\beta, \rho_\beta : \eta \leqslant \beta < \theta]$. This completes the proof.

Proof of Theorem 3.1. Existence of R and K follows from Lemmas 3.3 and 3.5. The assertion about cardinality is an immediate consequence of the cardinality restriction on the moid constructed in Lemma 3.3. This finishes the proof.

4. In this section, we consider the following situation.

$$\begin{aligned} K &- a \text{ skew field} \\ R &= K[x_\beta, \rho_\beta : 0 < \beta < \alpha] \\ \bar{R}_\beta &= K[x_\gamma, \rho_\gamma : 0 < \gamma < \beta] \\ R_\beta &= \bar{R}_\beta[x_\beta, \rho_\beta] \\ \rho_\beta(\bar{R}_\beta) &\subseteq K \qquad \text{for} \qquad 0 < \beta < \alpha. \end{aligned}$$

We shall maintain this notation throughout this section. It was shown in §3 that the above situation can be realized for every ordinal $\alpha > 1$.

THEOREM 4.1. *R is a pli-domain.*

Proof. Follows from Theorem 2.10.

PROPOSITION 4.2. *If q is a nonconstant monic monomial in the standard form in $\{x_\beta : 0 < \beta < \alpha\}$ then Rq is a nonzero proper two-sided ideal of R.*

Proof. We have

$$
\begin{aligned}
x_\beta x_\gamma &= \rho_\beta(x_\gamma)x_\beta && \text{if} \quad \gamma < \beta \\
&= x_\beta{}^2 && \text{if} \quad \gamma = \beta \\
&= x_\beta[\rho_\gamma(x_\beta)]^{-1}x_\gamma x_\beta && \text{if} \quad \gamma > \beta.
\end{aligned}
$$

Also, if $a \in K$ then

$$x_\beta a = \rho_\beta(a)x_\beta .$$

It follows that Rx_β is a nonzero proper ideal of R. The proof is completed by observing that if Ry and Rz are two-sided ideals then $Ry \cdot Rz = Ryz$ is also a two-sided ideal of R.

LEMMA 4.3. *The elements* $1 + x_\beta$ $(0 < \beta < \alpha)$ *are right linearly independent over* R.

Proof. Suppose n is the least positive integer for which there exists a non-trivial relation

$$\sum_{i=1}^{n} (1 + x_{\beta_i})\, r_i = 0 \tag{6}$$

where $0 < \beta_1 < \cdots < \beta_n$ and $r_i \in R^*(i = 1,\ldots, n)$. Let γ be the highest ordinal that occurs in the standard forms of r_i, $i = 1,\ldots, n$, and l be the highest degree to which x_γ occurs. Assume that (6) has the least value of γ and for such *a* γ the least value of l.

If $\gamma > \beta_n$, we can regard r_i as elements of $\bar{R}_\gamma[x_\gamma, \rho_\gamma]$ and put them in the form

$$r_i = r_i' x_\gamma + r_i''$$

where $r_i' \in R_\gamma$ and $r_i'' \in \bar{R}_\gamma$ for $i = 1,\ldots, n$. Thus, (6) becomes

$$\left\{ \sum_{i=1}^{n} (1 + x_{\beta_i})\, r_i' \right\} x_\gamma + \sum_{i=1}^{n} (1 + x_{\beta_i})\, r_i'' = 0.$$

Regarding this as an equation in $\bar{R}_\gamma[x_\gamma, \rho_\gamma]$, we get

$$\sum_{i=1}^{n} (1 + x_{\beta_i})\, r_i' = 0 = \sum_{i=1}^{n} (1 + x_{\beta_i})\, r_i'' .$$

Minimality of γ and l now implies $r_i' = r_i'' = 0$ and thus $r_i = 0$ for $i = 1,\ldots, n$. This contradicts the assumed nontriviality of (6).

If $\gamma < \beta_n$ then (6) becomes

$$\left\{\sum_{i=1}^{n-1} (1 + x_{\beta_i}) r_i + r_n\right\} + \rho_{\beta_n}(r_n) x_{\beta_n} = 0.$$

Regarding this as an equation in $R_{\beta_n}[x_{\beta_n}, \rho_{\beta_n}]$ and equating coefficients of x_{β_n}, we get $\rho_{\beta_n}(r_n) = 0 = r_n$ (since ρ_{β_n} is monomorphism.) As this contradicts minimality of n, we must have $\gamma = \beta_n$.

Let

$$r_i = \sum_{j=0}^{m_i} p_{ij} x_\gamma{}^j \qquad (i = 1,..., n)$$

where $p_{ij} \in \bar{R}_\gamma$ and $p_{im_i} \neq 0$. If $m_i > m_n + 1$ for at least one i, $i = 1,..., n-1$, then equating the coefficients of the highest power of x_γ in (6) we get a non-trivial relation of the form (6) containing fewer than n terms. As this possibility is rulled out, we must have $m_i \leqslant m_n + 1 = m$ say, for $i = 1,..., n-1$. Putting $p_{im} = 0$ if $m_i < m$ for $i = 1,..., n-1$, and equating coefficients of $x_\gamma{}^m$ in (6), we get

$$\sum_{i=1}^{n-1} (1 + x_{\beta_i}) p_{im} + \rho_{\beta_n}(p_{n\,m-1}) = 0. \tag{7}$$

Since $p_{n\,m-1} \neq 0$ and ρ_{β_n} is a monomorphism in K, $u = \rho_{\beta_n}(p_{n\,m-1})$ is a unit in R. It is easy to see that (7) gives the following relation:

$$(1 + x_{\beta_1})\{1 + p_{1m}u^{-1}(1 + x_{\beta_1})\} + \sum_{i=2}^{n-1} (1 + x_{\beta_i}) p_{im}u^{-1}(1 + x_{\beta_1}) = 0. \tag{8}$$

Since (8) has fewer than n terms, it must be a trivial relation so that

$$p_{1m}u^{-1}(1 + x_{\beta_1}) = -1.$$

Hence $1 + x_{\beta_1} \in U(R)$. However it is clear that $U(R) = K^*$ so that $1 + x_{\beta_1} \notin U(R)$. This completes the proof.

We recall a definition. A ring Λ (with unity) is a left (resp. right) primitive ring if there exists a maximal left (resp. right) ideal I of Λ such that (0) is the only two-sided ideal of Λ contained in I.

THEOREM 4.4. *If α is a limit ordinal then R is right primitive but not left primitive.*

Proof. Let l be any nonzero left ideal of R. By Theorem 4.1, $l = Ra$ for some $a \in R^*$. As α is a limit ordinal, there exists an ordinal $\lambda < \alpha$ which follows every ordinal that occurs as a subscript in the standard form of a. Thus $a \in \bar{R}_\lambda$ and $x_\lambda = [\rho_\lambda(a)]^{-1} x_\lambda a$. Thus $Rx_\lambda \subseteq l$. By Proposition 4.2, Rx_λ is a twosided ideal of R. Hence R is not a left primitive ring.

Let $A = \sum_{0<\beta<\alpha} (1 + x_\beta) R$. If $A = R$ then there exist $0 < \beta_1 < \cdots < \beta_n < \alpha$ and $r_i \in R^*$ $(i = 1,..., n)$ such that

$$\sum_{i=1}^{n} (1 + x_{\beta_i}) r_i = 1.$$

Thus,

$$(1 + x_{\beta_1})\{r_1(1 + x_{\beta_1}) - 1\} + \sum_{i=2}^{n} (1 + x_{\beta_i}) r_i(1 + x_{\beta_1}) = 0.$$

As $r_1(1 + x_{\beta_1}) \neq 1$, this contradicts Lemma 4.3. Thus A is a proper right ideal of R. Let I be a maximal right ideal containing A; such a right ideal exists by a routine application of Zorn's lemma. Suppose I contains a nonzero two-sided ideal T of R. Then, as shown above, there exists an ordinal λ $(0 < \lambda < \alpha)$ such that $Rx_\lambda \subseteq T$. Now x_λ and $1 + x_\lambda$ are both in I so that $I = R$, a contradiction. Hence R is right primitive. This completes the proof.

Let S denote the subset of all those elements of R which contain a nonzero element of K in their standard form. It is easy to see that, in R, product of two monomials remains a monomial when reduced to its standard form and that the product is an element of K^* if and only if both monomials are elements of K^*. These observations together with the uniqueness of standard form (Theorem 2.6) show that S is a monoid in R.

THEOREM 4.5. *S is a left localizable monoid in R. Every element in R^* can be expressed in the form sq where $s \in S$ and q is a monic monomial in the standard form.*

Proof. Let $s \in S$ and $r \in R^*$. As R is a pli-domain, $Rs + Rr = Rf$ for some $f \in R^*$. Consequently, $f = a_1 s + b_1 r$; $s = af$; $r = bf$ for some $a, a_1, b, b_1 \in R$. These give

$$\left.\begin{aligned} (1 - aa_1) s &= ab_1 r, \\ ba_1 s &= (1 - bb_1) r. \end{aligned}\right\} \tag{9}$$

If $a \notin S$ then clearly $aa_1 \notin S$ so that $(1 - aa_1) s \in S$ and $ab_1 r \notin S$. As this contradicts the first of equations (9), we have $a \in S$. Now if $b_1 \in S$ then $ab_1 \in S$

and the first of equations (9) gives a left common multiple and if $b_1 \notin S$ then $1 - bb_1 \in S$ and the second of equations (9) gives a left common multiple. Hence S is a left localizable monoid in R.

Now, let $r \in R^*$ be arbitrary. If, in the standard form, r has a nonzero term in K then $r \in S$. Otherwise, the standard form of r is $r = \sum_{i=1}^{n} q_i$, where q_i are all non-constant monomials. Put $q_i = q_i' x_{\beta_i}$ $(i = 1,..., n)$ where q_i' are monomials in the standard form containing indeterminates whose subscripts do not follow β_i. Let λ be the least of β_i, $i = 1,...., n$. If $\beta_i > \lambda$ then we have

$$x_{\beta_i} = [\rho_{\beta_i}(x_\lambda)]^{-1} x_{\beta_i} x_\lambda .$$

It follows that

$$r = \left\{ \sum_{\beta_i = \lambda} q_i' + \sum_{\beta_i > \lambda} q_i' [\rho_{\beta_i}(x_\lambda)]^{-1} x_{\beta_i} \right\} x_\lambda . \tag{10}$$

Clearly, the length of the monomial $q_i'[\rho_{\beta_i}(x_\lambda)]^{-1} x_{\beta_i}$ in its standard form equals the length of the monomial $q_i' x_{\beta_i} = q_i$ if $\beta_i > \lambda$. It follows that the sum of the lengths of the monomials $q_i (i = 1,..., n)$ is strictly greater than the sum of the lengths of the monomials in the cofactor of r in (10). An induction is thus available. This completes the proof.

We recall some definitions. A ring Λ is a *local* ring if Λ has a unique maximal left ideal. A domain Λ is a left (resp. right) *fir* if every left (resp. right) ideal is free and Λ has the invariant basis property. (See [6] for details.) A domain Λ is a (noncommutative) unique factorization domain if every nonzero element of Λ has a factorization as a product of a finite number of irreducible elements and a few things more (relating uniqueness which we shall not need.) See [5] for details. Let Λ be a domain and $a \in \Lambda^*$; dim a is the supremum of the lengths of chains of left ideals in the interval $[\Lambda a, \Lambda]$.

THEOREM 4.6. *Let $P = R_S$. Every left ideal of P is a two-sided ideal and has the form Pq where q is a monic monomial in the standard form. The set of two-sided ideals of P is well-ordered under reverse inclusion. P is a local pli-domain. If J denotes the Jacobson radical of P then $J = Px_1$ and $J^\beta \supseteq Px_\beta$ for $0 < \beta < \alpha$. If $\alpha > 2$ then P contains elements of infinite dimension, P is a left fir but not a right fir and P is not a unique factorization domain.*

Proof. By Theorem 4.5, S is a left localizable monoid in R so that $P = R_S$ exists. As R is a pli-domain, it can be easily seen that P is also a pli-domain. Using Theorem 4.5, it follows that every nonzero left ideal of P has the form Pq where q is a monic monomial in the standard form.

Let $0 < \lambda < \alpha$ and $s \in S$. Let $s = a + \sum_{i=1}^{n} q_i$ be the standard form of s, where $a \in K^*$ and q_i $(i = 1,..., n)$ are nonconstant monomials in the standard form. If $q_i \in \bar{R}_\lambda$ for some i then $x_\lambda q_i = \rho_\lambda(q_i)x_\lambda$ where $\rho_\lambda(q_i) \in K^*$. If $q_i \notin \bar{R}_\lambda$ then the last indeterminate in q_i is x_μ with $\lambda \leqslant \mu < \alpha$. Using

$$x_\mu = [\rho_\mu(x_\lambda)]^{-1}x_\mu x_\lambda$$

if $\lambda < \mu$, we can express $x_\lambda q_i$ as $q_i' x_\lambda$ where q_i' is a nonconstant monomial in the standard form. We thus have

$$x_\lambda s = \left[\rho_\lambda(a) + \sum_{i=1}^{n} q_i'\right] x_\lambda$$

where the constant term of $\rho_\lambda(a) + \sum_{i=1}^{n} q_i'$ is $\rho_\lambda(a + \sum_{q_i \in \bar{R}_\lambda} q_i)$. As ρ_λ is a monomorphism in K, this terms is nonzero. Thus, $\rho_\lambda(a) + \sum_{i=1}^{n} q_i'$ is an element of S. It follows that $x_\lambda s^{-1} \in Px_\lambda$ for every $s \in S$. As Rx_λ is a two-sided ideal of R by Proposition 4.2, Px_λ is a twosided ideal of P for every λ $(0 < \lambda < \alpha)$. It is now clear that every left ideal of P is a two-sided ideal (cf. proof of Proposition 4.2).

Let

$$q_1 = x_{\alpha_1}^{l_1} \cdots x_{\alpha_k}^{l_k}$$

and

$$q_2 = x_{\beta_1}^{m_1} \cdots x_{\beta_n}^{m_n}$$

be nonconstant monic monomials in the standard forms. If $\alpha_k < \beta_n$ then

$$x_{\beta_n} = [\rho_{\beta_n}(q_1)]^{-1} x_{\beta_n} q_1$$

shows that $Pq_2 \subseteq Pq_1$. Similarly $\alpha_k > \beta_n$ implies $Pq_2 \supseteq Pq_1$. If $\alpha_k = \beta_n$ and $l_k < m_n$ then

$$x_{\beta_n} = [\rho_{\beta_n}(x_{\alpha_1}^{l_1} \cdots x_{\alpha_{k-1}}^{l_{k-1}})]^{-1} x_{\beta_n} x_{\alpha_1}^{l_1} \cdots x_{\alpha_{k-1}}^{l_{k-1}}$$

shows that $q_2 \in Px_{\beta_n}^{l_k+1} \subseteq Pq_1$ so that $Pq_2 \subseteq Pq_1$. If $\alpha_k = \beta_n$ and $l_k > m_n$ then $Pq_2 \supseteq Pq_1$. An induction is now available to show that the set of two-sided ideals of P is totally ordered under reverse inclusion viz. $I_1 < I_2$ if and only if $I_1 \supsetneq I_2$. Since P is a pli-domain, it has the maximum condition on left ideals. Thus the minimum condition holds when the set of ideals is ordered under reverse inclusion. The set of twosided ideals is thus well-ordered under reverse inclusion.

It is clear from above that Px_1 is the unique maximal left ideal of P so that $J = Px_1$ and P is a local pli-domain.

Let β be an ordinal, $0 < \beta < \alpha$. Suppose $J^\gamma \supseteq Px_\gamma$ holds for all $0 < \gamma < \beta$. If β is a nonlimit ordinal, say $\beta = \lambda + 1$, then

$$J^\beta = J \cdot J^\lambda \supseteq Px_1 \cdot Px_\lambda = Px_1 x_\lambda .$$

Now,

$$x_\beta = [\rho_\beta(x_1 x_\lambda)]^{-1} x_\beta x_1 x_\lambda$$

shows that $Px_\beta \subseteq Px_1 x_\lambda$ so that $Px_\beta \subseteq J^\beta$. If β is a limit ordinal then

$$J^\beta = \bigcap_{0<\gamma<\beta} J^\gamma \supseteq \bigcap_{0<\gamma<\beta} Px_\gamma \supseteq Px_\beta .$$

This completes the transfinite induction and shows that $J^\beta \supseteq Px_\beta$ for $0 < \beta < \alpha$.

If $\alpha > 2$, then $x_2 = [\rho_2(x_1{}^n)]^{-1} x_2 x_1{}^n$ for every $n \in Z^+$ shows that x_2 is of infinite dimension in P. It also shows that x_2 has no factorization in terms of irreducible elements so that P is not a unique factorization domain. Since P is a pli-domain, it is clear that every left ideal of P is free. As P has a left quotient skew field it follows [*6*] that P has the invariant basis property. Thus P is a left fir. It now follows (Theorem 2.8 of [*6*]) that P is not a right fir. This completes the proof.

Lemma 4.7. *If* $0 < \gamma < \beta < \lambda < \alpha$ *then*

$$[\rho_\lambda(x_\gamma)]^{-1}\, x_\lambda R \subsetneq [\rho_\lambda(x_\beta)]^{-1}\, x_\lambda R$$

and

$$[\rho_\lambda(x_\gamma)]^{-1}\, x_\lambda P \subsetneq [\rho_\lambda(x_\beta)]^{-1}\, x_\lambda P.$$

Proof. We shall prove both the inclusions simultaneously. Observe that R_λ is also a pli-domain and therefore has a left quotient skew field, say $\underset{\sim}{Q}_\lambda$. Theorem 2.1 shows that $\rho_\lambda : R_\lambda \to K$ can be uniquely extended to a monomorphism $\tilde{\rho}_\lambda : \underset{\sim}{Q}_\lambda \to K$. Let

$$a = [\rho_\lambda(x_\beta)]^{-1} x_\lambda [\rho_\beta(x_\gamma)]^{-1} x_\beta$$

which is an element of R and also of P. Considering a as an element of $\underset{\sim}{Q}_\lambda$, we get

$$\begin{aligned} a &= \tilde{\rho}_\lambda(x_\beta^{-1})\, \tilde{\rho}_\lambda([\rho_\beta(x_\gamma)]^{-1})\, \tilde{\rho}_\lambda(x_\beta)\, x_\lambda \\ &= (\tilde{\rho}_\lambda\{x_\beta^{-1}[\rho_\beta(x_\gamma)]^{-1}\, x_\beta\})\, x_\lambda \\ &= \tilde{\rho}_\lambda(x_\gamma^{-1})\, x_\lambda = [\rho_\lambda(x_\gamma)]^{-1}\, x_\lambda . \end{aligned}$$

This establishes both the inclusions. If either of the inclusions is an equality then

$$[\rho_\lambda(x_\beta)]^{-1} x_\lambda = [\rho_\lambda(x_\gamma)]^{-1} x_\lambda y$$

where y is an element of R or P as the case may be. We then have

$$\begin{aligned} x_\lambda &= \rho_\lambda(x_\beta)[\rho_\lambda(x_\gamma)]^{-1}x_\lambda y \\ &= \rho_\lambda([\rho_\beta(x_\gamma)]^{-1}x_\beta x_\gamma)[\rho_\lambda(x_\gamma)]^{-1}x_\lambda y \\ &= \rho_\lambda([\rho_\beta(x_\gamma)]^{-1})\rho_\lambda(x_\beta)x_\lambda y \\ &= x_\lambda[\rho_\beta(x_\gamma)]^{-1}x_\beta y \end{aligned}$$

Cancelling x_λ from both sides we get

$$1 = [\rho_\beta(x_\gamma)]^{-1}x_\beta y.$$

Thus $x_\beta \in U(R)$. As $U(R) = K^*$, this is a contradiction. This proves the lemma.

The following two theorems are due to B.L. Osofsky ([*24*], [*25*]).

THEOREM 4.8. *If* $|R| \leqslant \aleph_n$ *then*

$$|\,\text{l.gl.dim}\, R - \text{r.gl.dim}\, R\,| \leqslant n + 1.$$

THEOREM 4.9. *Let* M_R *be a right* R*-module such that* $a \in M$, $r \in R$ *and* $ar = 0$ *implies either* $a = 0$ *or* $r = 0$. *Let* $M = \bigcup_{i\in I} a_iR$ *where* $\{a_iR : i \in I\}$ *form a totally ordered chain under inclusion. Let* Ω_n *be the first ordinal of cardinality* $\aleph_n$ *and put* $\Omega_{-1} = 1$. *Then, for* $n \geqslant -1$,

$$pd(M) = n + 1$$

if and only if the set of all ordinals $<\Omega_n$ *is cofinal in the set* $\{a_iR : i \in I\}$ *under inclusion.*

We are now in a position to prove the following theorem.

THEOREM 4.10. *If* $1 \leqslant n \leqslant \infty$, *there exists a pli-domain* D *such that* r.gl.dim $D = n$. *D can be taken to be a local pli-domain.*

Proof. By Theorems 3.1 and 4.1, for every ordinal $\alpha > 1$, there exists a pli-domain R with $|R| = \aleph_0|\alpha|$. It is easy to see that for the corresponding local pli-domain P, we have $|R| = \aleph_0|\alpha|$. Choose $\alpha = \Omega_n + 1$. As R and P are both pli-domains but not skew fields,

$$\text{l.gl.dim}\, R = \text{l.gl.dim}\, P = 1.$$

Thus, the cardinality restriction on R and P and Theorem 4.8 imply that

$$\text{r.gl.dim}\, R \leqslant n + 2$$

and

$$\text{r.gl.dim}\, P \leqslant n + 2.$$

However, taking $\lambda = \Omega_n$ in Lemma 4.7 and applying Theorem 4.8, R and P have a right ideal whose homological dimension is $n+1$. By the global dimension theorem [*1*], right global dimensions of R and P must be at least $n+2$. This completes the proof.

COROLLARY. *If* $1 \leqslant m \leqslant n \leqslant \infty$ *then there exists a left Noetherian domain* D *with left global dimension* m *and right global dimension* n.

Proof. If $m = n = \infty$, any nonregular local commutative Noetherian domain will work. If m is finite, take the domain Λ given by Theorem 4.10 with r.gl.dim $\Lambda = n - m + 1$ and let $D = \Lambda[t_1, \ldots, t_{m-1}]$ be the ring in the $m - 1$ commuting indeterminates over D. By a well-known theorem of Kaplansky, [*21*] it follows that l.gl.dim $D = m$ and r.gl.dim $D = n$. Since Λ is a pli-domain, by Hilbert basis theorem, D is a left Noetherian domain. This completes the proof.

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Theorem 4.10 was proved by the author for domains P; that the same proof works for domains R (due to an improvement of her result) was observed by Prof. B. L. Osofsky.

REFERENCES

1. AUSLANDER, M. On the dimension of modules and algebras. III. Global dimension. *Nagoya Math. J.* **9** (1955), 67–77.
2. BERGMAN, G. M. A ring primitive on the right but not on the left. *Proc. Am. Math. Soc.* **15** (1964), 473–475.
3. BOURBAKI, N. "Éléments de Mathématique; Algébre Commutative," Chapitre 2. Hermann, Paris, 1961.
4. CARTAN, E. AND EILENBERG, S. "Homological Algebra." Princeton Univ. Press, Princeton, New Jersey, 1956.
5. COHN, P. M. Noncommutative unique factorization domains. *Trans. Am. Math. Soc.* **109** (1963), 313–331.
6. COHN, P. M. Free ideal rings. *J. Algebra* **1** (1964), 47–69.
7. COHN, P. M. Hereditary local rings. *Nagoya Math. J.* **27** (1966), 223–230.
8. COHN, P. M. Torsion modules over free ideal rings. *Proc. London Math. Soc.* (3) **17** (1967), 577–599.
9. CURTIS, C. W. A note on noncommutative polynomials. *Proc. Am. Math. Soc.* **3** (1952), 965–969.

10. GOLDIE, A. W. Rings with maximum condition. Multigraphed lecture notes. Yale University, 1964.
11. HALL, MARSHALL, JR. "The Theory of Groups." Macmillan, New York, 1959.
12. HERSTEIN, I. N. Topics in ring theory. Lecture notes; University of Chicago, 1965.
13. HERSTEIN, I. N. A counter-example in Noetherian rings. *Proc. Nat. Acad. Sci. USA* **54** (1965), 1036–1037.
14. JACOBSON, N. The radical and semi-simplicity for arbitrary rings. *Am. J. Math.* **67** (1945), 300–320.
15. JACOBSON, N. Structure of rings. Am. Math. Soc. Coll. Publ. Vol. 37 (revised edition), 1964.
16. JATEGAONKAR, A. V. An example in principal left ideal rings. Unpublished.
17. JATEGAONKAR, A. V. Left principal ideal domains. *J. Algebra* **8** (1968), 148–155.
18. JENSEN, C. U. On the homological dimension of rings with countably generated ideals. *Math. Scand.* **18** (1966), 97–105.
19. JOHNSON, R. E. Unique factorization in principal right ideal domains. *Proc. Am. Math. Soc.* **16** (1965), 526–528.
20. KAPLANSKY, I. On the dimension of modules and algebras. X. A right hereditary ring which is not left hereditary. *Nagoya Math. J.* **13** (1958), 85–88.
21. KAPLANSKY, I. Homological dimension of rings and modules. Multigraphed lecture notes. University of Chicago, 1959.
22. LESIEUR, L. AND CROISOT, R. Extension au cas non commutatif d'un théorème de Krull et d'un lemme d'Artin-Rees. *J. Reine Angew. Math.* **204** (1960), 216–220.
23. ORE, O. Theory of non-commutative polynomials. *Ann. Math.* **34** (1933), 480–508.
24. OSOFSKY, B. L. Upper Bounds on Homological Dimensions. *Nagoya Math. J.* **32** (1968), 315–323.
25. OSOFSKY, B. L. Homological dimension and the continuum hypothesis. *Trans. Am. Math. Soc.* **132** (1968), 217–230.
26. SIERPINSKI, W. Cardinal and ordinal numbers. Polska Acad. Nauk, Monografie Math. Vol. 34. Warszawa, 1958.
27. SMALL, L. W. An example in Noetherian rings. *Proc. Nat. Acad. Sci. USA* **54** (1965), 1035–1036.
28. SMALL, L. W. Hereditary rings. *Proc. Nat. Acad. Sci. USA* **55** (1966), 25–27.
29. ZARISKI, O. AND SAMUEL, P. "Commutative Algebra," Vol. 1. Van Nostrand, New York.